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A Comparative Study on Reliability Analysis of Cohesive Soil Slope using Subset Simulation and Other Methods

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Abstract
In geotechnical engineering, uncertainties arise due to variation in loads, soil characteristics, ground stratification and so on. Reliability analysis based on probabilistic approaches is particularly suitable to deal with such uncertainties. In this paper, a newly developed reliability analysis method, namely Subset simulation (SS) method has been implemented to study the stability of a cohesive slope. The results of reliability analysis obtained from SS method are also compared with three other methods, namely First order second moment method (FOSM), First order reliability method (FORM) and Direct Monte-Carlo simulation (MCS) method. The various reliability models have been used in a spreadsheet environment using MS-Excel. The developed spreadsheet-based platform implementing all four methods contains two common models i.e. deterministic model and the uncertainty model. The SS method uses another model called uncertainty propagation using subset simulation (UPSS) in addition to the two above-mentioned models. The factor of safety (FS) of the slope is determined using ordinary method of slices under undrained condition. The probability of failure (P_f) and its corresponding reliability index (β) of the proposed slope has been determined using all four methods. A software called Geo-Studio (SLOPE/W) has been used to tally the results of reliability analysis of the slope considered herein. The results obtained from the different methods show that the SS method gives better performance in terms of efficiency and resolution especially at low failure probability i.e., $P_f < 0.001$. Also, the SS method helps in identifying the significant depth where the most probable critical slip surface is located.

Author Keywords. Reliability Index, Probability of Failure, FOSM, FORM, Direct, MCS SS.

Type: Research Article

1. Introduction

During analysis of any geotechnical structure such as footing, slopes, embankment etc., it is of utmost importance to consider the effect of spatial variability of soil properties. Baecher and Christian (2005) studied that inherent variability cannot be minimized as these are independent, therefore, classified this phenomenon as aleatory uncertainty in nature. Besides the inherent spatial variability, several other uncertainties (e. g. measurement uncertainty, statistical uncertainty and transformation model uncertainty) are also an integral part of the task of estimation of soil properties during geotechnical characterization work. As pointed out by Kulhawy and Trautmann (1996), the measurement uncertainties arise due to device errors, data handling and testing errors etc. The measurement uncertainty can be minimized by improving the knowledge on test procedures and equipment, therefore, classified as epistemic uncertainty (Baecher and Christian 2005). Kulhawy and Trautmann (1996) considered statistical uncertainty as a part of measurement uncertainty, which may arise due to availability of limited test data. Phoon and Kulhawy (1999) established the correlation between design property and measured property of soil by using transformational models. The inability of the model to represent the system's true conditions results into transformational model uncertainty. These uncertainties can be reduced if proper correlations between the relevant parameters can be established. Kulhawy et al. (2006), Fenton and Griffiths (2002, 2003), Fenton, Griffiths, and Williams (2005), and Schuëller (2007) experienced that the epistemic uncertainties did not affect the response of geotechnical structures but the inherent spatial variability of soil contribute significantly to the response of geotechnical structures. Thus, the affected estimations of soil characteristics and ground stratification influence the design (or analysis) of geotechnical structures significantly. In slope stability problems, various uncertainties are usually encountered, such as variation in pore water pressure, missing soil exploration data, testing errors, estimation of soil properties, which cannot be reproduced accurately.

In deterministic method of analysis, FS is defined as the ratio of resisting moment (M_R) to overturning moment (M_O). The slope is considered to be safe, when the obtained FS value exceeds unity. In reliability analysis, FS is expressed in terms of its mean and variance. Ang and Tang (1984), Phoon (2008), and Sivakumar Babu and Mukesh (2004) studied that the probability of failure (P_f) and reliability index (β) can be used to address the uncertainties in soil including the inherent spatial variability of soil for assessing the performance of geotechnical analysis and design process. In recent decades, various probabilistic methods have been developed to calculate the value of β and P_f for geotechnical structures specially for slope, such as FORM (Baecher and Christian 2005; Low and Tang 1997, 2007), FOSM (Baecher and Christian 2005; Tang, Yucemen, and Ang 1976; Christian, Ladd, and Baecher 1994; Husein Malkawi, Hassan, and Abdulla 2000; Wu and Kraft 1970; Cornell 1972; Hasofer and Lind 1974) and direct MCS method (Ditlevsen 1981; Hammersley and Handscomb 1964; Robert and Casella 2004; El-Ramly, Morgenstern, and Cruden 2002; Low 2003). El-Ramly, Morgenstern, and Cruden (2002) and Low (2003) have shown that these probabilistic methods use probabilistic estimations of soil characteristics and ground stratification as key input data and return the value of P_f and/or β as an output. As compared to FORM or FOSM method, the direct MCS method is getting more popularity in assessing the complex geotechnical structures due to its robustness and conceptual simple technique to calculate the failure probability. However, direct MCS method faces lack of efficiency and resolution at low probability levels. Also, this method does not always ensure the generation of sample data in the failure region. To overcome the above drawback of direct MCS method, a practical method for performing the reliability analysis of slope has been presented by Wang, Cao, and Au (2011), Au, Cao, and Wang (2010), and Cao, Wang, and Li (2016) which is known as Subset simulation (SS) method, also called an advanced version of MCS method. The SS method uses Markov Chain Monte Carlo Simulation (MCMCS) (Hastings 1970) technique to generate the sample sets based on the Bayes' theorem of conditional probability. The SS method ensures generation of sample failure points at low failure probability levels, which is not always possible with direct MCS method. The MCMCS technique is based on the Metropolis algorithm (Metropolis et al. 1953).

The Geo-Studio software (SLOPE/W 2007) has the capability of performing probabilistic stability analysis of slope based on direct MCS technique. This software can carry out stability analysis of slopes using both limit equilibrium technique as well as Finite Element Method

(FEM). Sivakugan and Das (2009) have used Geo-studio software for different types of slope stability problems and recommended it for solving slope stability problems.

This paper presents a reliability analysis of a finite cohesive slope based on the probabilistic approach. The analysis has been performed using FOSM, FORM, direct MCS and SS method by considering the spatial variability of undrained shear strength of soil (S_u) over the domain of interest. The FS of the slope is determined using ordinary method of slice (Fellenius 1936). The saturated unit weight (γ_{sat}) of soil, undrained shear strength (S_u) of soil, the x and y coordinates of the centre of the slip surface and the radius of the slip surface (r) have been used to prepare sample space data. The slope stability analysis model has been developed in a MS-Excel spreadsheet which mainly consists of three parts, namely deterministic model, uncertainty model and uncertainty propagation using subset simulation. It has been observed that SS method has significantly improved the efficiency and resolution of simulation as compared to other methods especially at low probability levels. Moreover, SS method significantly reduces the number of samples to be generated by direct MCS to reach the desired level of accuracy. Also, this method ensures the generation of sample data in the failure region which is not guaranteed in direct MCS method.

2. Methods

The FS of the cohesive slope under drained condition is calculated using ordinary method of slices as per Equation (1) as follows:

$$FS = \frac{\sum S_u \, \Delta L}{\sum W \sin \alpha} \tag{1}$$

where S_u = undrained shear strength of soil, ΔL = length of arc, W = weight of slice, α = inclination of slice base. The probability of failure (P_f) of the slope is defined as the probability for which the minimum factor of safety value is less than one (i.e. $P_f = P(FS < 1)$. The reliability index (β) and P_f of a slope are correlated to each other and can be expressed as per Equation (2) (Baecher and Christian 2005; Wang, Cao, and Au 2011; Au, Cao, and Wang 2010; Cao, Wang, and Li 2016).

$$P_f = 1 - \Phi(\beta) = \Phi(-\beta) \tag{2}$$

where Φ is the standard normal cumulative distribution function of Gaussian random variable. The β and P_f can be estimated by various methods as illustrated below.

2.1. First Order Second Moment Method (FOSM)

The FOSM is a relatively simple method for uncertainty quantification. It is based on the first-order Taylor's series expansion. The β can be calculated using FOSM as per Equation (3) (Ang and Tang 1984; Baecher and Christian 2005; Cornell 1972; Cao, Wang, and Li 2016).

$$\beta = \frac{\mu_{FS} - 1}{\sigma_{FS}} \tag{3}$$

2.2. First Order Reliability Method (FORM)

The FORM is formulated by transforming the basic random variables, $x = [x_1,\ x_2, \dots, x_n]^T$ which defines the limit state function, $g(x)$ into uncorrelated standard Gaussian variables, $\mathrm{x} = [x_1, x_2, \dots, x_n]^T$. The β is defined as a measure of the shortest distance between the origin of failure region and the design point in *n*-dimensional space and expressed as $\beta = \sqrt{\mathrm{x}^{*T}\mathrm{x}^*}$ where x^* is design point, which is defined in standard Gaussian space as the nearest point on the limit state surface ($g_1(\mathrm{x})$=0) from the origin in failure region. The probability of failure approximated at design point $\mathrm{x}^*$is computed from reliability index β as $P_f \approx \int_{F_1} \phi(\mathrm{x}) d\mathrm{x} = \Phi(-\beta)$, where F_1 represents the linear failure region. The β can be expressed in matrix formulation (Veneziano 1974; Ditlevsen 1981; Madsen, Krenk, and Lind 2006) as per Equation (4).

$$\beta = \underbrace{min}_{g_1(\mathrm{x})=0} \sqrt{\left[\frac{\mathrm{x}_i - \mu_i}{\sigma_i}\right]^T [\overline{\mathrm{R}}]^{-1} \left[\frac{\mathrm{x}_i - \mu_i}{\sigma_i}\right]} \qquad i = 1, 2, \dots, n \tag{4}$$

in which μ_i is a vector representing the mean value of uncertain variable, σ_i is mean vector representing standard deviation of uncertain variable and $[\overline{\mathrm{R}}]^{-1}$ is the inverse of the correlation matrix of the uncertain variables, $[(\mathrm{x}_i - \mu_i)/\sigma_i]$ is vector of n uncertain variables transformed into standard Gaussian space and $g_1(\mathrm{x})$ is limit state function.

2.3. Direct Monte Carlo Simulation Method (MCS)

Direct MCS is a numerical method of continuously calculating an empirical or mathematical operator, containing a random variable of known probability distribution. The result obtained from each repetition is considered as true data, which is analogous to the data observed during the experiment. Figure 1 illustrates the systematic flowchart of MCS method for performing the slope stability analysis. According to Robert and Casella (2004), to obtain the expected performance in P_f, the number of samples used in direct MCS should be at least equal to $10/P_f$ which translates to considering a minimum sample size of 10,000 for obtaining P_f level of 0.001.

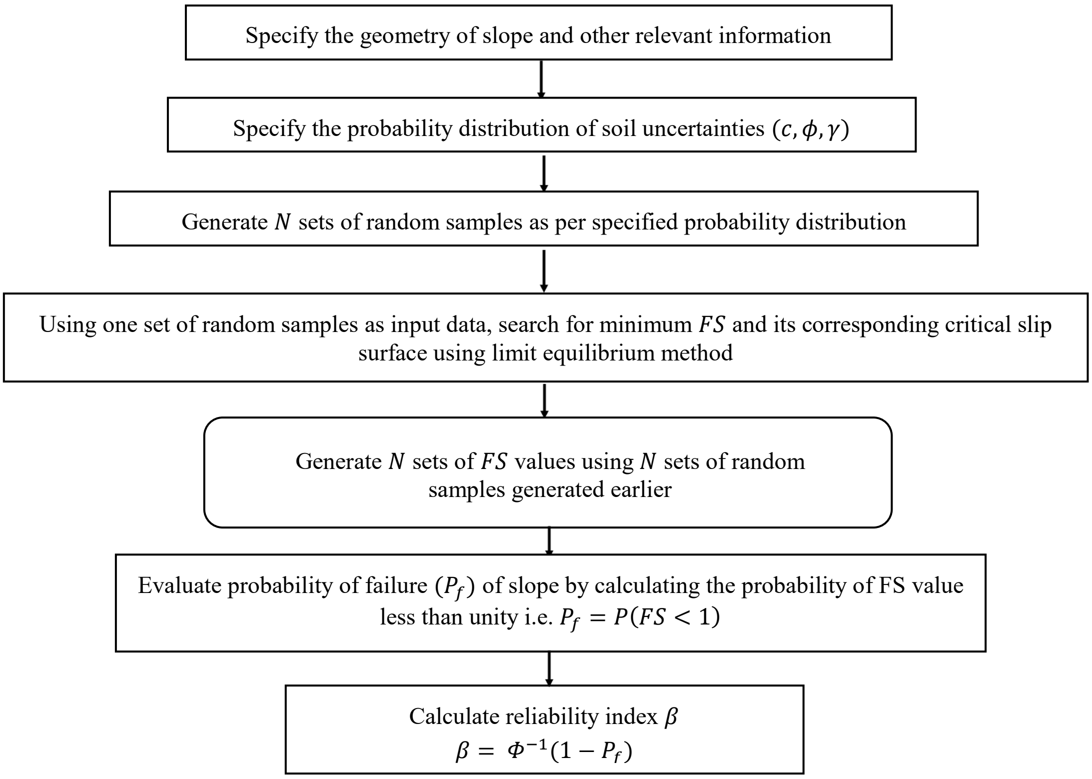


Figure 1: Systematic representation of Direct Monte Carlo simulation for slope stability analysis

2.4. Subset Simulation (SS)

The main problem associated with Direct MCS is to estimate the small failure probabilities. For small P_f value, the coefficient of variance (c.o.v.) can be written as per Equation (5) as follows:

$$c.o.v = \sqrt{\frac{1-P_f}{P_f\, n}} \sim \frac{1}{\sqrt{P_f\, n}}\; P_f \rightarrow 0 \tag{5}$$

As the probability of failure (P_f) diminishes, the c.o.v. increases dramatically. Hence, for small failure probabilities or rare events, direct MCS method is not efficient enough and therefore an advanced MCS method known as Subset simulation method has been introduced to improve the efficiency and resolution of MCS (Wang, Cao, and Au 2011; Au, Cao, and Wang 2010; Au, Ching, and Beck 2007; Au and Beck 2001; Au and Wang 2014). The SS method uses Bayes' conditional probability theory and Markov Chain Monte Carlo simulation (MCMCS) (Hastings 1970) technique based on metropolis algorithm (Metropolis et al. 1953) to efficiently compute small failure probability. This method is generated from a fact that small failure probability event can be stated as sequence of intermediate failure probability events with larger conditional failure probability, hence, converting a rare failure event into sequence of more frequent ones.

Let $Y = FS$ be the critical response for slope stability problem and the probability that Y greater than any threshold value y is of interest, i.e. $P(Y = FS > y)$ and $y = y_m > y_{m-1} > \ldots > y_2 > y_1$ be decreasing order of m intermediate threshold value. Let $E_i = Y > y_i, i = 1, 2, \ldots, m$ be the intermediate events. The failure probability of an event E, $P(E) = P(Y > y)$ can be written as $P(E) = P(E_m) = P(E_m E_{m-1} \ldots E_1)$. Using product rule of probability, $P(E)$ can be rewritten as $P_f = P(Y > y) = P(Y > y_m) = P(Y > y_1)P(Y > y_2|Y > Y_1) \times \cdots \times P(Y > y_m|Y > y_{m-1})$.

During implementation, the threshold values $(y_m, \ldots, y_2, y_1)$ are generated in such a way that the sample estimate of $P(E_1)$ and $\{P(E_i|E_{i-1}), i = 1, 2, \ldots, m\}$ corresponds to a specific value of conditional probability P_0. Wang, Cao, and Au (2011), Au, Cao, and Wang (2010), Au, Ching, and Beck (2007), Au and Beck (2001), and Au and Wang (2014) have recommended the value of $P_0 = 0.10$ as a good choice. The SS procedure for generating samples of FS on condition that $\{\mathrm{Y} = FS > \mathrm{y_i},\ \mathrm{i} = 1, 2, \ldots, \mathrm{m}\}$ corresponding to specified target probability $P(Y > y_i)$ is described next.

Firstly, N samples of independent and identically distributed (i.i.d.) random variables as original probability distribution function (PDF) are generated by direct MCS. Then, the response $Y$of the corresponding N Samples are computed and ranked in ascending order. The $(1 - P_0) \times N^{th}$ value in the ascending list of Y $\{Y_k, k = 1, 2, \ldots, N\}$ is taken as the value of $y_1$such that the sample estimate for $P(E_1) = P(Y > y_1)$ is P_0. In other words, we can say that, there is $P_0 N$ samples generated from direct MCS have failure event, $E_1 = Y > y_1$. Now, using these $P_0 N$ samples as seeds for MCMCS, N additional conditional samples are simulated having $E_1 = Y > y_1$. Now the previous $P_0 N$ seed samples are discarded. The Y values of N additional conditional samples obtained are again ranked into ascending order and $(1 - P_0) \times N^{th}$ value is taken as y_2 having $E_2 = Y > y_2$. Again, the sample $P_0 N$ having $E_2 = Y > y_2$ are used as seed in MCMC to generate another N additional conditional sample having $E_2 = Y > y_2$. This procedure is repeated m times until $E_m = Y > y_m$ is achieved. The subset simulation contains $(m + 1)$ steps, also referred as $(m + 1)$ level as it includes one level of direct MCS to generate unconditional sample and additional m steps of MCMCS to simulate conditional samples (Wang, Cao, and Au 2011; Au, Cao, and Wang 2010; Au, Ching, and Beck 2007; Au and Beck 2001; Au and Wang 2014). The total number of samples generated from $m + 1$ level is equal to $N + m\,(1 - P_0)\,N$.

3. Illustrative Example

A simple homogeneous cohesive slope considered by Au, Cao, and Wang (2010), have been taken in this study to assess its reliability having uncertainty in undrained shear strength. The cross-section and soil properties of slope are shown in Figure 2. The ordinary method of slices is used to assess the stability of the slope under undrained condition. A circular slip surface is assumed having center coordinate (x_c, y_c) and with radius r. The hard stratum is assumed to be present at 15 m below top of the cohesive soil.

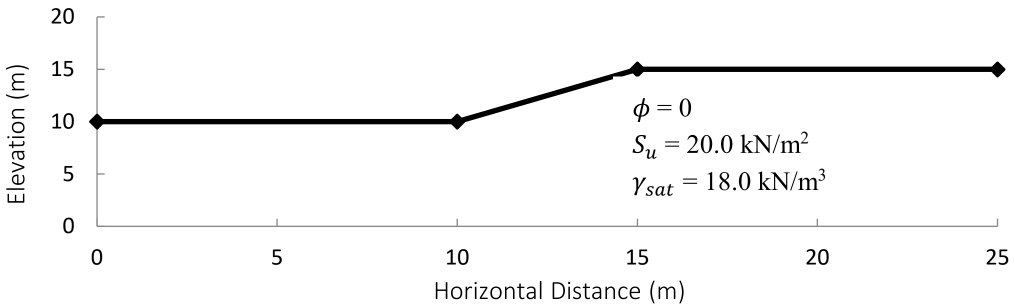


Figure 2: Cross-section for slope stability problem

4. Methodology

The reliability analysis of the cohesive soil slope has been performed using the following methodology as discussed below.

4.1. Input variables

The critical slip surface has been obtained by choosing the different combination of (x, y) coordinates of the centre of slip surface and the radius (r) of the slip surface to obtain minimum factor of safety, in a sequence from a global coarser grid to a local denser grid.

The spatial variation in undrained shear strength of soil (S_u) along the vertical direction is modeled by one-dimensional random field theory. The S_u at same elevation are considered to be fully correlated. The undrained shear strength with depth is log-normally distributed having an exponential correlation. Let $S_u(d)$ be the undrained shear strength at any depth (d), then the correlation between $ln[S_u(d_i)]$ and $ln[S_u(d_j)]$ at depth d_i and d_j is given as per Equation (6)

$$\bar{R} = R_{ij} = e^{\left(-\frac{2|d_i - d_j|}{\lambda}\right)} \quad (6)$$

in which λ is correlation length. Correlation length is defined as the length up to which the soil parameters are fully correlated. In other words, it can be stated that within the correlation length, the soil properties remain same at every point. Therefore, it can be said that $ln[S_u(d_i)]$ and $ln[S_u(d_j)]$ are effectively uncorrelated when $|d_i - d_j| \geq \lambda$, whereas $ln[S_u(d_i)]$ and $ln[S_u(d_j)]$ are highly correlated when $|d_i - d_j| \ll \lambda$ (Vanmarcke 1977, 2010). The γ_{sat} of soil is taken as constant of 18.0 kN/m^3. Table 1 shows the critical slip surface, material properties and their variation used in the analysis.

Input Variable	Distribution Type	Values
Coordinate (x_c, y_c) and radius (r) of critical slip surface	Deterministic	(2.6, 8.8) & 16 m
Undrained shear strength (S_u)	Lognormal	Mean = 20 kN/m^2 c.o.v. = 20 % λ = 2 m
Saturated unit weight (γ_{sat})	Deterministic	18 kN/m^3

Table 1: The values of input variables and their distribution

4.2. Deterministic model

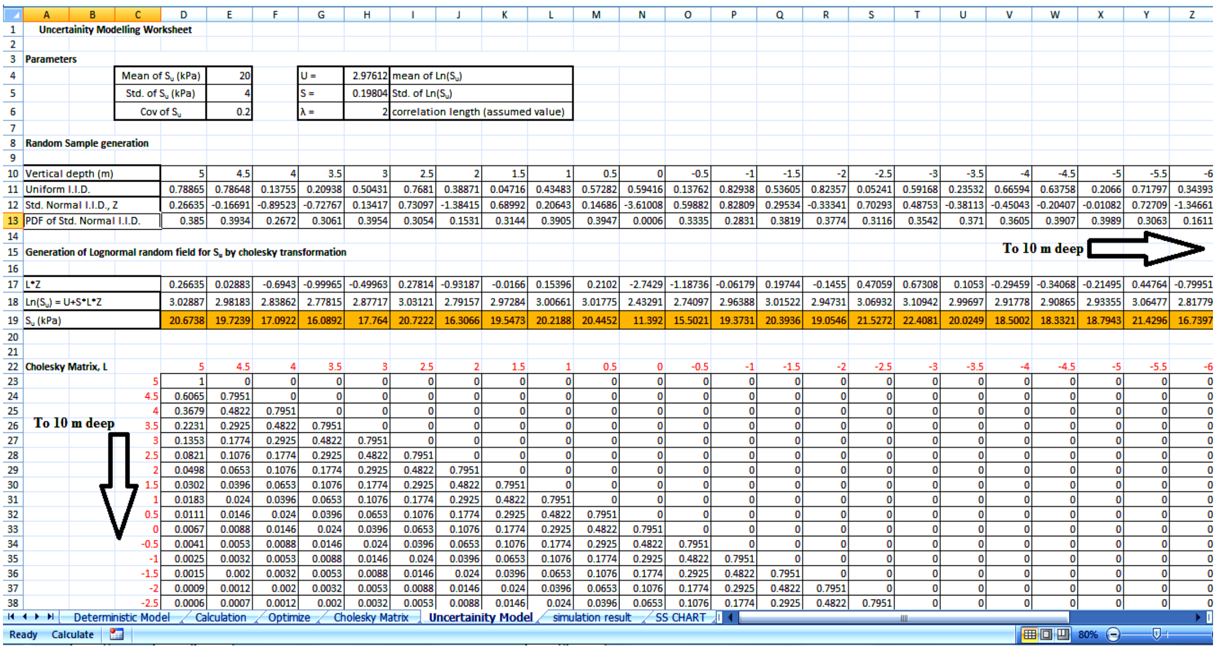
The implementation of MCS based reliability analysis (or design) is carried out in a spreadsheet using MS-Excel, by a package of worksheets and Visual Basic for Application (VBA) functions. The framework is mainly divided into three forms, namely deterministic model, uncertainty model and uncertainty propagation model. Deterministic modelling is the procedure of evaluating the FS for a given set of system parameters such as slope geometry, soil profile, soil characteristics slip surface geometry etc. using limit equilibrium methods. No concept of probability is used in generating the deterministic model and it can be easily generated without any knowledge about probabilistic (or reliability) analysis. VBA codes have been written for determining the ratio of resisting moment (M_R) to driving moment (M_D) with respect to different values of (x_c, y_c) and (r) and then identifying the minimum value as the FS and its corresponding critical slip surface. For the present problem, the value of FS is obtained as 1.248, which corresponds to the critical slip surface having coordinate (2.6, 8.8) and radius of 16.0 m.

4.3. Uncertainty model

An uncertainty model is used to generate the uncertain parameters which are considered as random variables in reliability based analysis (or design). Based on the detail information of random variable (e.g. distribution type, correlation details and statistics), random samples of the random variables are generated. For the example problem, S_u of soil is considered as uncertain parameter with depth. The 15 m deep cohesive soil is assumed to be divided into 30 layers of 0.50 m thickness. Total thirty-one uniformly i.i.d. random variables are required at 31 depths. Let $\bar{S} = [S_u(d_1), S_u(d_2), \dots, S_u(d_n)]^T$ be vector of (S_u) at depth $d_1, d_2, \dots, d_n$. According to Ang and Tang (1984), Au, Cao, and Wang (2010), and Wang, Cao, and Au (2011), in space domain when (S_u) is log-normally distributed, it can be represented as per Equation (7)

$$\bar{S} = \exp(\mu\bar{I} + \sigma\bar{L}\,\bar{Z}) \quad (7)$$

where μ and σ are the mean and standard deviation of $ln[S_u(d)]$, $\bar{I}$ is a column vector with n components having all elements equal to unity, $\bar{Z}$ is standard Gaussian vector with n dimension, $\bar{L}$ is a $n \times n$ dimensional lower triangular matrix obtained by Cholesky decomposition of correlation matrix $\bar{R}$ as stated in Eq. 9, such that $\bar{R} = \bar{L}\bar{L}^T$. For this problem, the dimension n equals to 31. Figure 3 shows the uncertainty model worksheet consisting four parts namely input parameters, random sample generation, generation of lognormal random field for S_u and generation of Cholesky matrix obtained using Matlab code. The values of S_u generated in the uncertainty model worksheet are random and therefore, are linked with the deterministic model worksheet to obtain the random samples of FS. The F9 key in Excel can also be used to generate the random samples of the FS. By doing so, one can easily perform FOSM, FORM and direct MCS by continuously pressing F9 key. Instead of continuously pressing F9 key, a VBA macro code has been run in MS-Excel to calculate the random values of FS at a time. For the current example problem, a total of 1850 samples of FS are generated and reliability analysis of the slope have been carried out using FOSM, FORM and direct MCS using Excel.



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	**Uncertainity Modelling Worksheet**																									
2																										
3	**Parameters**																									
4			Mean of S_u (kPa)		20		U =	2.97612	mean of Ln(S_u)																	
5			Std. of S_u (kPa)		4		S =	0.19804	Std. of Ln(S_u)																	
6			Cov of S_u		0.2		λ =	2	correlation length (assumed value)																	
7																										
8	**Random Sample generation**																									
9																										
10	Vertical depth (m)			5	4.5	4	3.5	3	2.5	2	1.5	1	0.5	0	-0.5	-1	-1.5	-2	-2.5	-3	-3.5	-4	-4.5	-5	-5.5	-6
11	Uniform I.I.D.			0.78865	0.78648	0.13755	0.20938	0.50431	0.7681	0.38871	0.04716	0.43483	0.57282	0.59416	0.13762	0.82938	0.53605	0.82357	0.05241	0.59168	0.23532	0.66594	0.63758	0.2066	0.71797	0.34393
12	Std. Normal I.I.D., Z			0.26635	-0.16691	-0.89523	-0.72767	0.13417	0.73097	-1.38415	0.68992	0.20643	0.14686	-3.61008	0.59882	0.82809	0.29534	-0.33341	0.70293	0.48753	-0.38113	-0.45043	-0.20407	-0.01082	0.72709	-1.34661
13	PDF of Std. Normal I.I.D.			0.385	0.3934	0.2672	0.3061	0.3954	0.3054	0.1531	0.3144	0.3905	0.3947	0.0006	0.3335	0.2831	0.3819	0.3774	0.3116	0.3542	0.371	0.3605	0.3907	0.3989	0.3063	0.1611
14																										
15	**Generation of Lognormal random field for S_u by cholesky transformation**																						**To 10 m deep**			
16																										
17	L*Z			0.26635	0.02883	-0.6943	-0.99965	-0.49963	0.27814	-0.93187	-0.0166	0.15396	0.2102	-2.7429	-1.18736	-0.06179	0.19744	-0.1455	0.47059	0.67308	0.1053	-0.29459	-0.34068	-0.21495	0.44764	-0.79951
18	Ln(S_u) = U+S*L*Z			3.02887	2.98183	2.83862	2.77815	2.87717	3.03121	2.79157	2.97284	3.00661	3.01775	2.43291	2.74097	2.96388	3.01522	2.94731	3.06932	3.10942	2.99697	2.91778	2.90865	2.93355	3.06477	2.81779
19	S_u (kPa)			20.6738	19.7239	17.0922	16.0892	17.764	20.7222	16.3066	19.5473	20.2188	20.4452	11.392	15.5021	19.3731	20.3936	19.0546	21.5272	22.4081	20.0249	18.5002	18.3321	18.7943	21.4296	16.7397
20																										
21																										
22	**Cholesky Matrix, L**			5	4.5	4	3.5	3	2.5	2	1.5	1	0.5	0	-0.5	-1	-1.5	-2	-2.5	-3	-3.5	-4	-4.5	-5	-5.5	-6
23			5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24			4.5	0.6065	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25			4	0.3679	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	**To 10 m deep**		3.5	0.2231	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27			3	0.1353	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28			2.5	0.0821	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29			2	0.0498	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30			1.5	0.0302	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31			1	0.0183	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32			0.5	0.0111	0.0146	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0	0
33			0	0.0067	0.0088	0.0146	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0	0
34			-0.5	0.0041	0.0053	0.0088	0.0146	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0	0
35			-1	0.0025	0.0032	0.0053	0.0088	0.0146	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0	0
36			-1.5	0.0015	0.002	0.0032	0.0053	0.0088	0.0146	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0	0
37			-2	0.0009	0.0012	0.002	0.0032	0.0053	0.0088	0.0146	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0	0
38			-2.5	0.0006	0.0007	0.0012	0.002	0.0032	0.0053	0.0088	0.0146	0.024	0.0396	0.0653	0.1076	0.1774	0.2925	0.4822	0.7951	0	0	0	0	0	0	0

Deterministic Model / Calculation / Optimize / Cholesky Matrix / **Uncertainity Model** / simulation result / SS CHART

Ready Calculate 80%

Figure 3: Uncertainty Model Worksheet

4.4. Uncertainty propagation

The uncertainty and deterministic model are linked together and SS process is invoked for uncertainty propagation as an Excel Add-In called Uncertainty Propagation using Subset Simulation (UPSS) (Au, Cao, and Wang 2010; Wang, Cao, and Au 2011; Cao, Wang, and Li 2016; Au and Wang 2014). After each simulation run, the UPSS gives the plot for driving variable ($i.e.,\ P(Y > y\ )$ versus threshold level y and based on the information, complementary cumulative density function (CCDF), histogram or probability of failure $\left(P_f\right)$ can be estimated. For the example problem, SS is performed using following parameters i.e., number of samples per level, N = 500, P_0 = 0.1 and number of simulation level = 2. The total number of samples generated is equal to 500 + 2 (1 – 0.1) 500 = 1400 (i.e., 500 sample in first level, 450 sample in second level and 450 sample in third level).

5. Results and Discussion

The Geo-studio software (SLOPE/W 2007) is implemented to perform the deterministic slope stability analysis of the slope shown in Figure 4. The soil properties of the slope have also been shown in the figure. The value of the FS using the software is obtained as 1.250. The radius of critical slip surface and its coordinates are found out to be 16.667 m and (2.1 m, 9.9 m) respectively as illustrated in Figure 11. Further, probabilistic slope stability analysis using direct MCS in Slope/W software is then performed, taking the value of correlation length (λ) equal to 2.0 m. The probability of failure $\left(P_f\right)$ and reliability index (β) calculated for 2000 samples are found out to be 0.65 and 2.48 respectively.

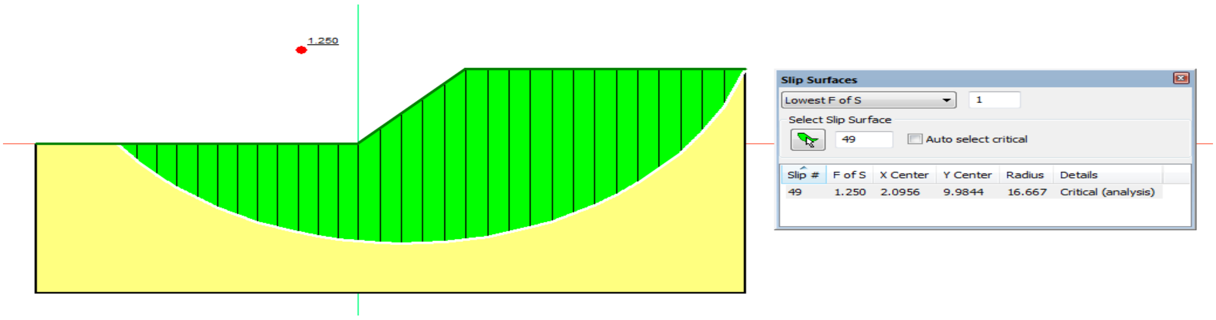


Figure 4: Results obtained from Geo-studio software (Slope/W)

Geo –Studio software is used to tally the result obtained from the MS-Excel spreadsheet. As seen in Table 2, the factor of safety (FS) and the radius of slip surface (r) obtained using Geo-Studio matches with those obtained using Excel sheet. Similar results were reported by Au, Cao, and Wang (2010) for the same example problem as shown in the table. However, there is some differences in the result obtained for the centre coordinate (x_c, y_c) of critical slip surface which may be due to the difference in the process of generating data samples. The results obtained in the present study varies up to 8 % compared to the results reported by Au, Cao, and Wang (2010).

Method	Factor of Safety (FS)	Radius of slip Surface, r (m)	co-ordinate of Slip Surface, x_c (m)	co-ordinate of Slip Surface, y_c (m)
Ordinary Method (MS-Excel Sheet)	1.248	16.0	2.6	8.8
Ordinary Method (Geo-studio)	1.250	16.67	2.1	9.9
Ordinary Method (Au, Cao, and Wang 2010)	1.250	16.0	2.4	9.2

Table 2: Comparison of critical slip surface and factor of safety

The possible range of coordinates (x, y) and radius (r) of slip surface obtained from Excel spreadsheet is shown in Table 3. As shown in table, the radius of slip surface has been taken in the range of 11.0 m to 16.0 m, x coordinate in the range of 1.0 m to 4.0 m and y coordinate in the range of 7.0 m to 10.0 m. Based on the data of (x, y) and (r), a grid is formed having interval of 1.0 m to identify the region with lowest factor of safety (FS) values. After obtaining the lowest FS region, the grid is further refined with a smaller interval of 0.2 m to identify the critical slip surface.

Parameter	Minimum	Maximum	Range
Coordinate x (m)	1	4	3
Coordinate y (m)	7	10	3
Radius r (m)	11	16	5

Table 3: Range of centre coordinates and radius of slip surfaces

Table 4 summarizes the results of direct MCS in Excel spreadsheet for λ = 2.0 m for different target FS values. A total of 1850 samples have been taken for direct MCS. For a specific target factor of safety (say 1.2), out of 1850 samples of factor of safety, 374 samples were found to have FS values less than 1.2. In other words, it can be stated that out of 1850 samples, 374 samples failed. Therefore, the probability of failure, P_f is calculated as 374/1850 = 20.22 %. This corresponds to β of ϕ^{-1} (79.78 %) = 0.83.

Effective Correlation Length λ (m)	Simulation Method	Number of Samples	Target FS	Number of Samples < Target FS	Probability of Failure, P_f (%)	Reliability Index (β)
2	Direct MCS	1850	1.1	51	2.76	1.92
		1850	1.2	374	20.22	0.83
		1850	1.3	977	52.81	-
		1850	1.4	1538	83.14	-
		1850	1.5	1764	95.35	-
		1850	1.6	1839	99.40	-
		1850	1.7	1848	99.89	-

Table 4: Summary of Direct MCS results

The histogram of the FS obtained from 1850 direct MCS samples have been shown in Figure 5. As shown in the figure, out of 1850 samples, 13 samples have factor of safety values less than one (i.e., FS < 1). The probability of failure (P_f), thus calculated is 13/1850 = 0.70% and the resolution of P_f obtained as 1/1850 = 0.054%. Such a low resolution may not be adequately sufficient for the P_f equals to 0.7 %.

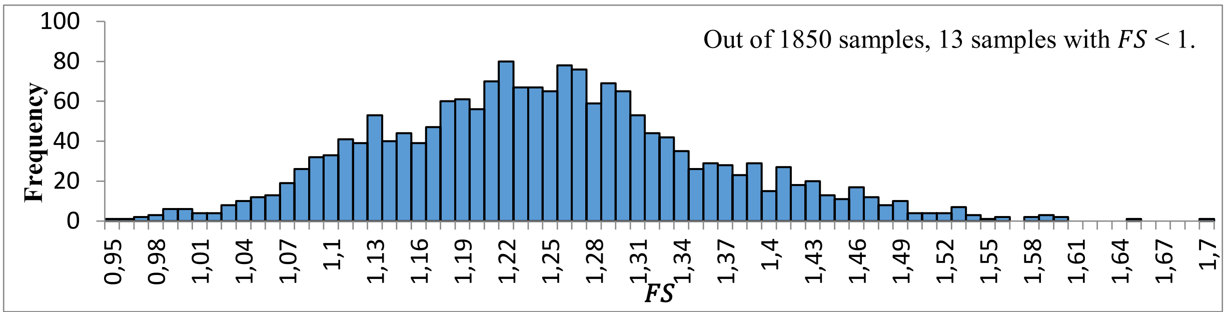


Figure 5: Factor of safety histogram from direct MCS

Figure 6 to Figure 8, shows the histogram of the FS obtained from three different levels of subset simulation. The subset simulation has been implemented for 500 + 450 + 450 = 1400 samples having initial value of P_0 = 0.1. The first level of subset simulation, also referred as level 0 is similar to direct MCS and the number of samples generated is equal to 500, out of which 3 samples have FS < 1, as illustrated in Figure 6. The 500 samples generated are then arranged in ascending order of their FS values and 50 samples (i.e., (P_0 =0.1) × 500) having lowest value of FS are used to generate another 450 samples (i.e., (1 - P_0 =0.1) × 500) in the second level or level 1 of subset simulation. As shown in Figure 7, thirty-seven samples out of 450 samples have FS < 1 and all the samples have relatively small factor of safety values and fall within the boundary of FS < 1.13. As compared to the direct MCS, the occurrence of failure events increases significantly during second level of subset simulation. Figure 8 shows the third level or level 2 of the simulation and further 307 samples out of 450 samples have FS < 1. The samples obtained during level 2, are found to be move further to low FS region (i.e., all the samples have their FS < 1.01). The probability of failure, P_f is calculated as 0.1 × 0.1 × 307/450 = 0.68 % and its corresponding reliability index (β) can be found out using equation 23 as ϕ^{-1}(0.32%) = 2.47. The resolution of P_f is calculated as 0.1 × 0.1 × 1/450 = 0.002%. Therefore, the resolution has increased significantly compared to the value 0.054% obtained in the case of direct MCS which also translates to higher efficiency. Thus, subset simulation is helpful in achieving desired level of accuracy with comparatively lesser number of samples at low probability levels in comparison with direct MCS. Also, it can be noticed that with increase in simulation levels, the samples generated by subset simulation are gradually shifting towards the failure region. Further, these samples are investigated to get the information about failure events as shown in Figure 9.

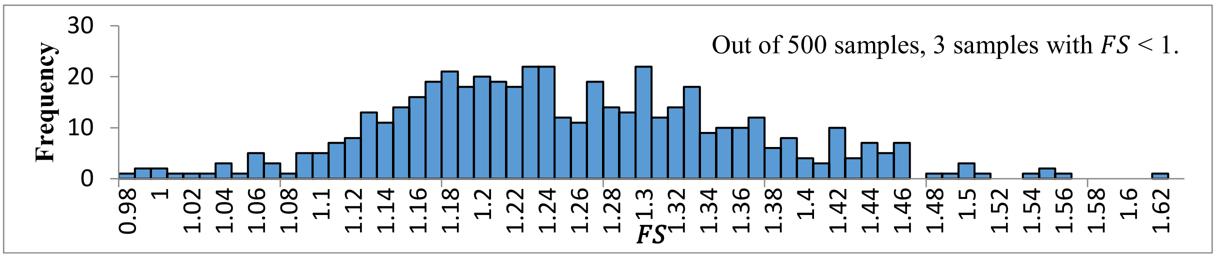


Figure 6: Factor of safety histogram from Subset simulation (First level or level 0)

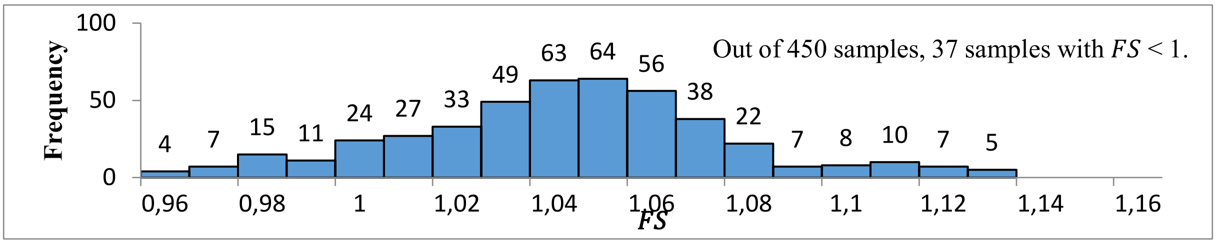


Figure 7: Factor of safety histogram from Subset simulation (Second level or level 1)

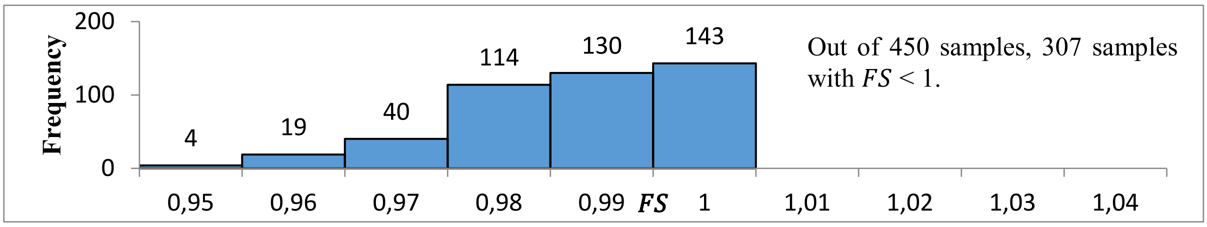


Figure 8: Factor of safety histogram from Subset simulation (Third level or level 2)

Figure 9 shows the average value of S_u of the soil at different depth and different simulation levels. It shows the full detailed information about the behaviour of the slope when failure occurs. As seen, the average value of S_u for soil layers between -7.5 m and 0.5 m decreases significantly and hence, it can be said that the S_u values of soil layer between -7.5 m and 0.5 m are critical from stability point of view as compared to the S_u values in the soil layer above 0.5 m depth and below -7.5 m depth. Therefore, the depth from -7.5 m and 0.5 m is identified as the significant depth as these depths are more susceptible to failure. The depth above 0.5 m and below -7.5 m is identified as insignificant depth as these depths are less susceptible to failure.

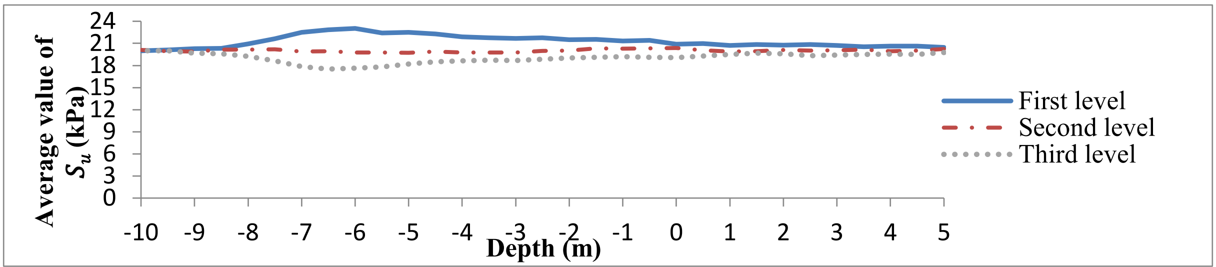


Figure 9: Average value of S_u at different depth

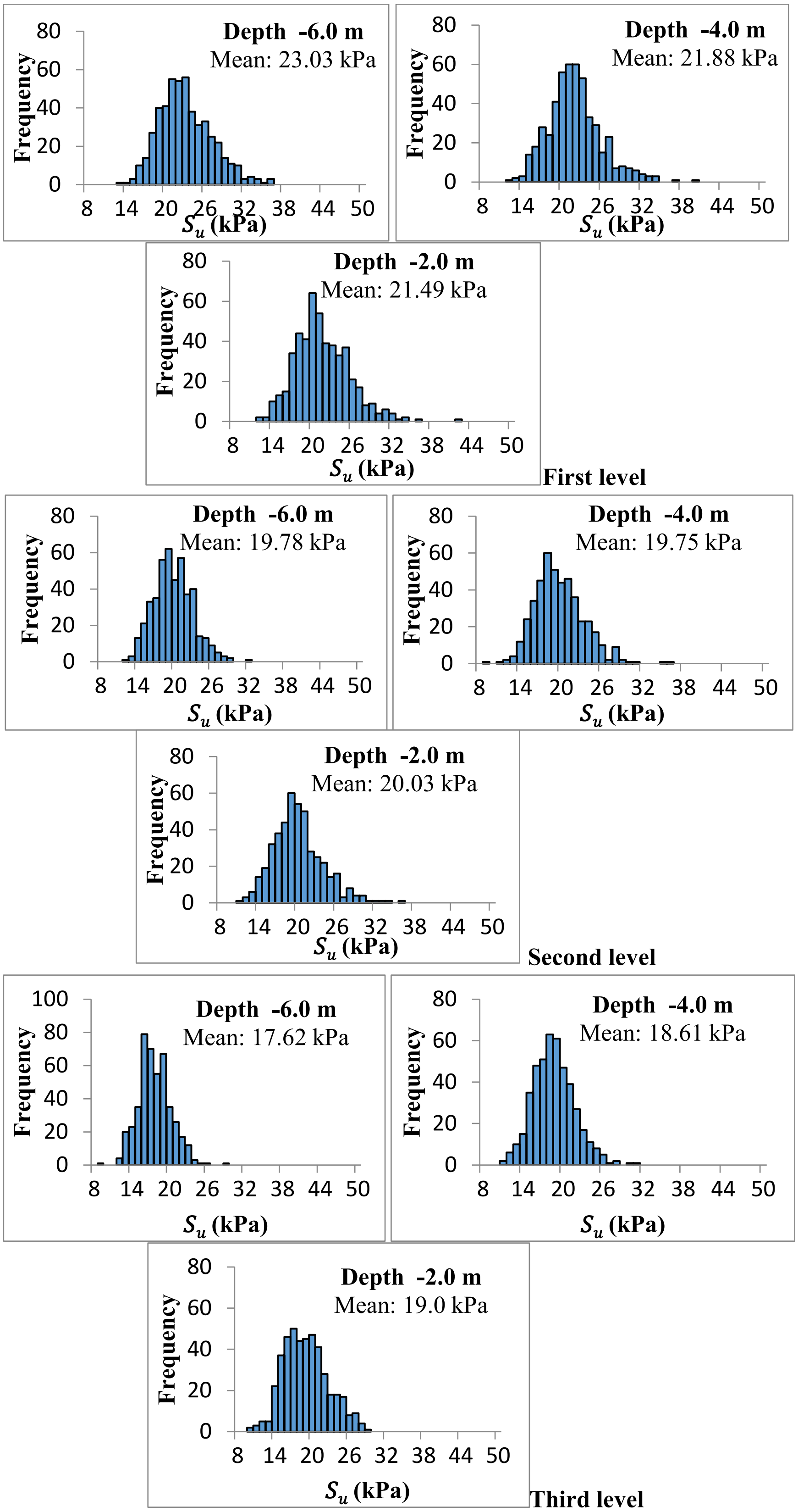


Figure 10: Distribution of S_u at significant depth

Figure 10 shows the histograms of the S_u at three significant depths i.e. -6.0 m, -4.0 m and -2.0 m for three different simulation levels. As seen, the distribution tends to shift towards the lower side with increase in the simulation levels. For comparison point of view, Figure 11 shows the histogram of the S_u at three insignificant depths, -8.0 m, -9.0 m and -10.0 m. As seen in the figure, the distribution of S_u is similar in all three-simulation level, thus representing the less sensitivity at depth more than -7.5 m.

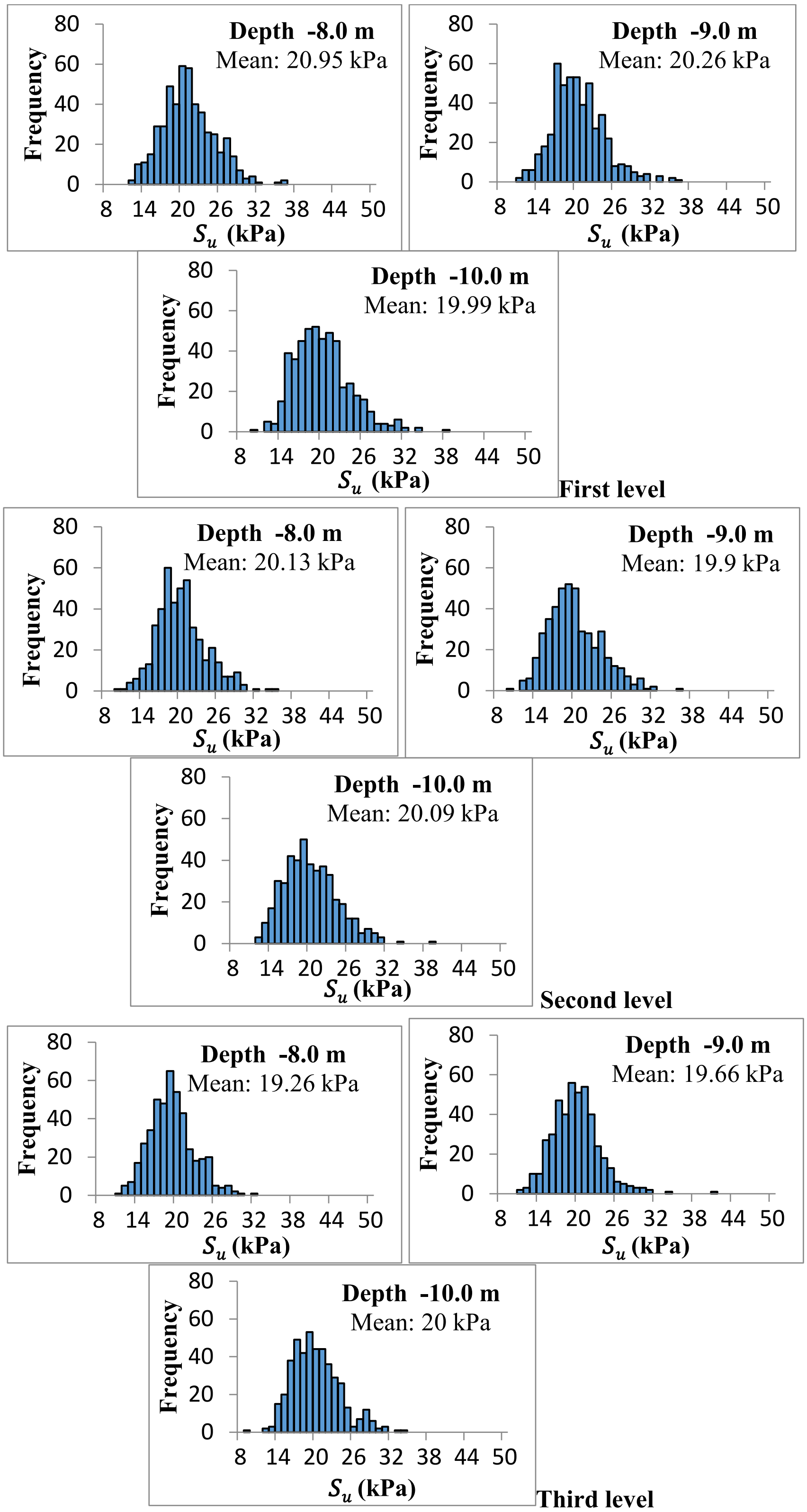


Figure 11: Distribution of S_u at insignificant depth

The FOSM has been applied to calculate the β from 1850 direct MCS samples of factor of safety generated using Excel spreadsheet. The corresponding critical slip surface has centre of coordinate (x_c, y_c) = (2.6 m, 8.8 m) and radius (r) = 16.0 m. The corresponding FS obtained is equal to 1.248. The mean and standard deviation of FS have been calculated for λ = 2.0 m and found out to be 1.249 and 0.097 respectively as shown in Figure 12. The reliability index (β) is calculated and found out to be 2.56. The probability of failure, P_f is calculated as $1 - \Phi(2.56)$ = 0.52 %.

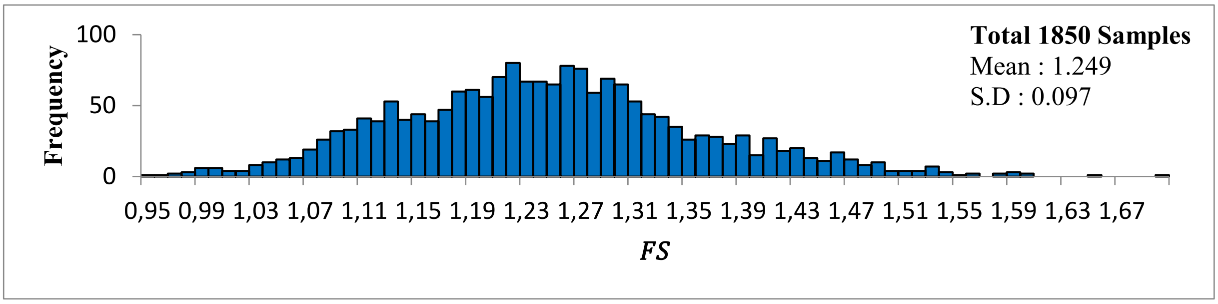


Figure 12: Factor of safety histogram showing mean and standard deviation of samples used in FOSM

The FORM analysis has been performed on 1850 samples of factor of safety generated using direct Monte-Carlo simulation in MS-Excel spreadsheet at 31 different depths. The reliability index β is calculated using Equation (4). Each term in the Equation (4) is computed and solved using code written in Matlab. The correlation matrix $[\overline{\mathrm{R}}]$ has dimension of 31 × 31 obtained using Equation (6) which is shown in Figure 13. The reliability index β is calculated for all 1850 samples of FS at different depths and the minimum of these values is reported as final reliability index. The reliability index obtained using FORM is 2.62 which corresponds to the probability of failure of 0.44 %. For total 1850 samples, an average value of $((\mathrm{x}_i - \mu_i)/\sigma_i)$ at different depth is shown in Figure 14. The histogram of the distribution of $((\mathrm{x}_i - \mu_i)/\sigma_i)$ at different depth is shown in Figure 15.

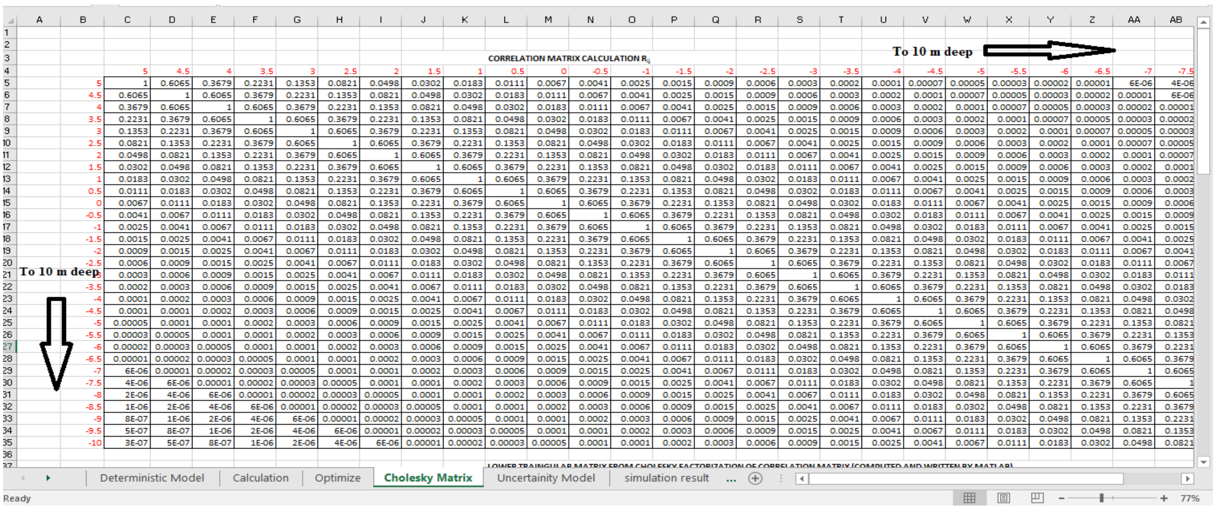


Figure 13: Correlation Matrix $[\overline{\mathrm{R}}]$

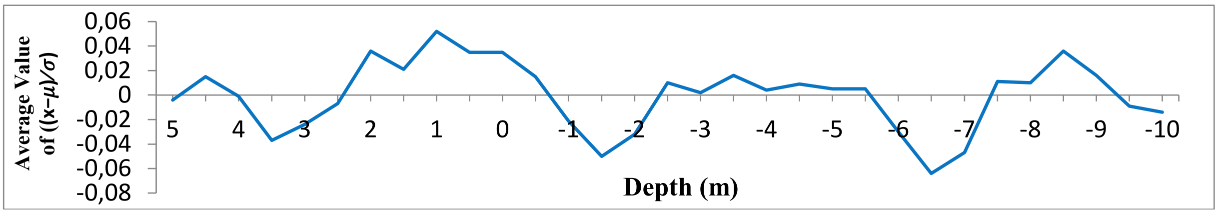
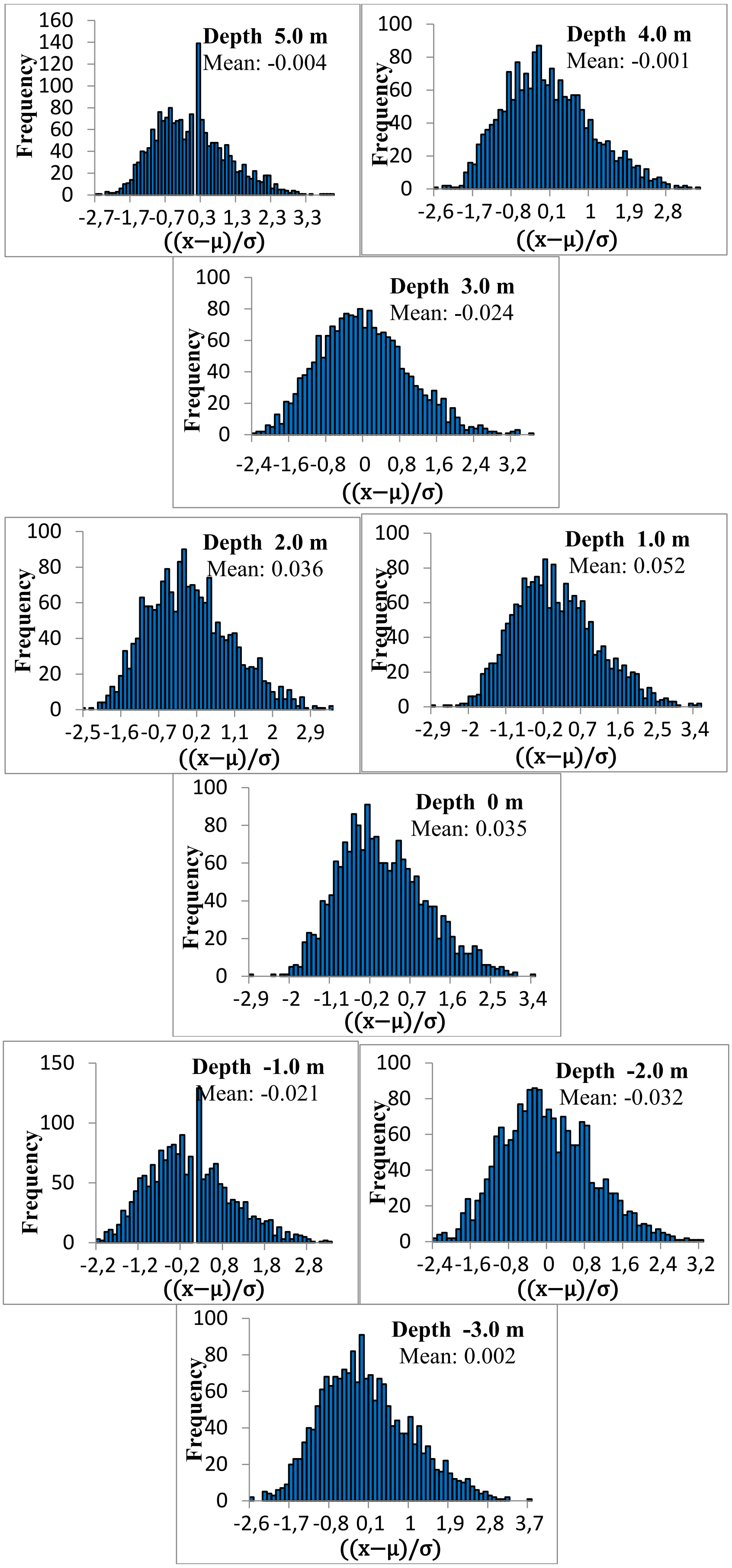


Figure 14: Average value of $((\mathrm{x}_i - \mu_i)/\sigma_i)$ at different depth for 1850 samples in FORM


Depth 5.0 m
Mean: -0.004
Depth 4.0 m
Mean: -0.001
Depth 3.0 m
Mean: -0.024
Depth 2.0 m
Mean: 0.036
Depth 1.0 m
Mean: 0.052
Depth 0 m
Mean: 0.035
Depth -1.0 m
Mean: -0.021
Depth -2.0 m
Mean: -0.032
Depth -3.0 m
Mean: 0.002
Frequency
((x−μ)/σ)

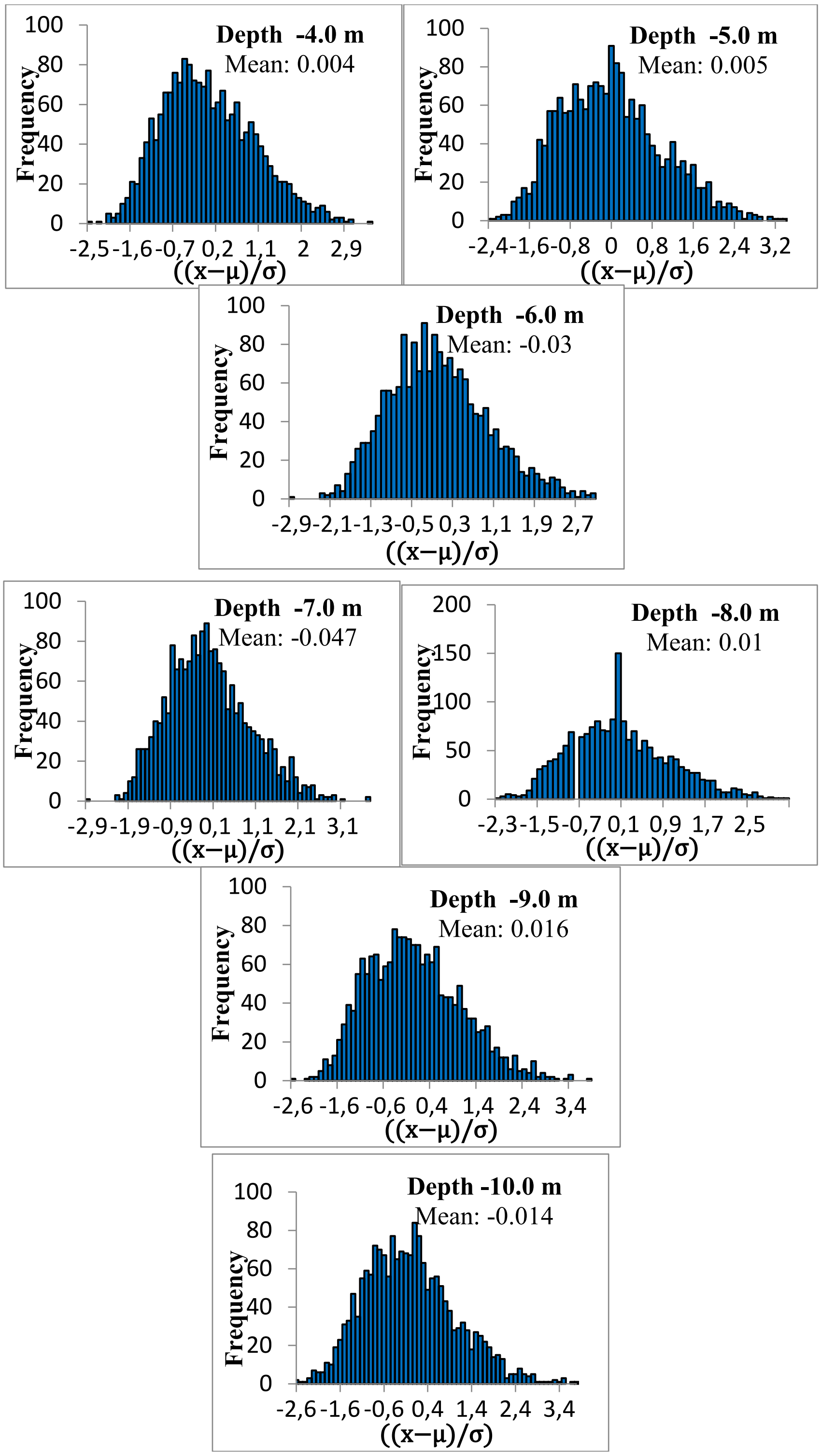


Figure 15: Histogram showing the distribution of $((x_i - \mu_i)/\sigma_i)$ at different depth for 1850 samples

Table 5 summarizes the results obtained from different reliability methods having correlation length equal to 2.0 m. Five different methods have been implemented to obtain the probability of failure P_f and reliability index β of the slope. The second column shows the number of samples of FS generated for different method. As seen in the table, the value of P_f varies from 0.44% to 0.70% having maximum relative difference among different methods is about 37 %. The reliability index of the slope varies from 2.46 to 2.62.

The FOSM method usually takes a fixed critical slip surface and does not consider uncertainties in slip surface therefore, underestimates the P_f value as reported by Wang, Cao, and Au (2011) and Cao, Wang, and Li (2016). This observation is presented in Table 5 as 26 % decrease in relative difference in P_f as compared to direct MCS (MS-Excel).

In FORM, the reliability index β is calculated using Excel spreadsheet and Matlab to obtain minimum distance of interest. The reliability index value is calculated from the critical slip surface obtained from a deterministic model worksheet having the value of the soil properties equal to their mean values. With the change in the soil properties or slip surface parameters, different β values have been obtained. As FORM assumes a linear failure domain, it overestimates the β value and thus, underestimates the uncertainty of failure. Table 5 shows the relative difference of 37% in P_f as compared to direct MCS (MS-Excel). Similar observation has been reported by Wang, Cao, and Au (2011) and Cao, Wang, and Li (2016) for slope stability problem.

The similar situation arises while performing direct MCS with Slope/W. It is a known fact that the correlation length is a measure of the spatial variability of the soil properties and therefore, the critical failure surface is also supposed to vary spatially with the change in correlation length. For the problem under consideration, a correlation length λ = 2.0 m has been considered which also accounts for the uncertainty of the soil properties. Slope/W uses a fixed critical slip surface and therefore, P_f values are underestimated compared to direct MCS (MS-Excel). The relative difference in P_f between direct MCS (Slope/W) and direct MCS (MS-Excel) is 7 % as shown in Table 5.

Reliability Method	**Number of Sample**	**Reliability Index (β)**	**Probability of failure (P_f) %**	**Relative difference in P_f (%)**	**Expected Performance level**
FOSM	1850	2.56	0.52	-26.0	Average
FORM	1850	2.62	0.44	-37.0	Below average
Direct MCS (MS-Excel)	1850	2.46	0.70	N/A#	N/A#
Direct MCS (Slope/W)	2000	2.48	0.65	-7.0	Good
Subset Simulation (MS-Excel)	1400	2.47	0.68	-3.0	Good

#Base value for calculating relative difference and performance level

Table 5: Summary of result obtained from different reliability method

6. Summary and Conclusions

The present study mainly focuses on the reliability analysis of a cohesive slope in MS-Excel spreadsheet environment based on the probabilistic approach. The effect of uncertainties arising due to the spatial variability of soil was examined. The comparative study on the results of slope analysis using different reliability methods such as FOSM, FORM, direct MCS using MS-Excel, direct MCS using software Geo-Studio (Slope/W) and an advanced MCS method called subset simulation in MS-Excel has been done. Based on the above results and discussion, the following conclusions can be drawn:

i. The factor of safety based on the deterministic approach using ordinary method of slices is found to be 1.248 and the critical slip surface has coordinates (2.6 m, 8.8 m) and radius of 16.0 m.
ii. The critical slip surface has been obtained by considering the possible range of the radius of slip surface (r) from 11.0 m to 16.0 m, the possible range of x coordinate of the slip surface from 1.0 m to 4.0 m and the possible range y coordinate of the slip surface from 7.0 m to 10.0 m by forming a grid.
iii. The direct MCS based reliability analysis has been performed with the help of deterministic model and uncertainty model by considering the uncertainty in the undrained shear strength (S_u) of soil at every 0.5 m depth. The effective correlation length equal to 2.0 m has been considered. Based on the sample generated from direct MCS method, FOSM and FORM are executed.
iv. The reliability index (β) obtained for 1850 samples using FOSM and FORM are 2.56 and 2.62 respectively. The corresponding probability of failure (P_f) is found to be 0.52% and 0.44% respectively.
v. Direct MCS using software Geo-Studio (Slope/W) has been performed for 2000 samples and the value of reliability index and its corresponding probability of failure is found to be 2.48 and 0.65% respectively.
vi. In direct MCS using Excel, a total 13 samples out of 1850 samples failed. The probability of failure is found to be 0.70% which corresponds to the reliability index of 2.46.
vii. Direct MCS is a very simple and efficient approach for performing the reliability analysis as compared to the FOSM and FORM.
viii. For small failure probability (or, rare events), the direct MCS method does not guarantee the generation of sample data in the failure region. Also, a very large number of samples are needed to be generated to get the desired level of accuracy in P_f.
ix. Subset simulation method has shown that the desired level of accuracy in P_f (i.e., 0.001) can be achieved with a very less number of samples i.e., 1400. Therefore, SS has significantly shown better performance in terms of the efficiency and resolution of simulation especially at low probability levels (i.e., $P_f < 0.001$).
x. In Subset simulation method, 3 samples failed out of 500 samples in first level of simulation, 37 samples failed out of 450 samples in second level of simulation and 307 samples failed out of 450 samples in third level of simulation. The probability of failure is found to be 0.68% which corresponds to the reliability index of 2.47.
xi. The resolution in P_f obtained from Subset simulation (i.e. 0.002%) has significantly increased as compared to the resolution obtained in the direct MCS (i.e. 0.054%).
xii. The Subset simulation analysis shows that the soil layer between -7.5 m to 0.5 m depth are more susceptible to failure as the values of undrained shear strength (S_u) between these layers decreases significantly. These soil layers play a crucial role in the slope stability analysis and hence termed as significant depth. The analysis also shows that the soil layer above 0.5 m depth and below -7.5 m depth are less sensitive to failure as the values of S_u in these layers are almost similar and hence termed as insignificant depth.
xiii. The study also demonstrates that the subset simulation method can help in understanding the nature of complex problem and better assess the risk involved in it. It will also help in guiding the geotechnical practitioners specially related to slope stability in their decision-making process.

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